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Extension of Cauchy Riemann System in Higher Dimensions

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Preface

So far, as we known, the theory of a holomorphic function has not only reached its fullness and beauty in terms of structure but also enriched many applications in different fields.
In the theory of partial differential equations sense, the theory of a holomorphic function is essentially the theory of the solution of the following Cauchy-Riemann system.

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = 0. \end{cases} \tag{1}$$

The real part and imaginary part of the holomorphic function $f(z) = u + iv$ are harmonic functions. But not with any two harmonic functions u and v, then $u + iv$ is a holomorphic function: They must be pairs of harmonic functions associated together by a specified rule (conjugate rule). Here, the conjugate rule is the Cauchy-Riemann condition.
The ideas of complex analysis started in the middle of the 18th century, first of all in connected with the Swiss mathematician, Leonhard Euler, and its mainly results in the 19th century have introduced by Augustin-Louis Cauchy, Georg Friedrich Bernhard Riemann and Karl Theodor Wilhelm Weierstrass.
As more and more new problems emerge from the realities that need to be solved, more research has been done to expand the Cauchy-Riemann system (which is also an extension of the theory of a holomorphic function). Looking back at these expansions, one can see that, the authors find several ways, linking the harmonic functions together.
As we known, to defined a holomorphic function in complex variable, there must be two harmonic functions which are adjoined together by the Cauchy-Riemann condition.
It neccessary to solved following problems: Is there any way to presented the Cauchy-Riemann condition in a succinct and concise way, directly to the function $f(z)$ does not depending on the real part and imaginary part?
Dimitrie D. Pompeiu was able to discover that way in 1912 when he proposed the

new definition about "aréolaire derivative" (surface derivative) (see [1a] and [20]).

$$\frac{\partial f}{\partial \overline{z}}(z_0) = \lim_{\gamma \to z_0} \frac{\frac{1}{2i}\int\limits_{\gamma} f(z)dz}{mesG} \tag{2}$$

where G is arbitrary domain which contains a point z_0 and the boundary of the domain G is a curve γ, $f(z)$ is continuously function. If $f \in C^1$ then from equation (2), we obtain:

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}),$$

and then, the Cauchy-Riemann condition equivalence if "aréolaire derivative" is vanished, that means:

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

This resuls is very important in investigation in complex analysis and special case of expand in other ways. There are many authors investigate about "aréolaire derivative" to solve particular problem, that be mentioned in Theodorescu ([3]), Angelescu ([4]), Nicolescu ([5]), Moisil ([6c,d]), etc.
The first and more natural extension of the theory of a holomorphic function of a complex variable is to constructed a holomorphic function theory in several complex variables. Essentially, that is a mapping

$$f : \Omega \subset \mathbb{C}^n \longrightarrow \mathbb{C},$$

and satisfying some neccessary conditions. Let $z_k = x_k + iy_k$, $k = 1, 2, ..., n$, $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $y = (y_1, ..., y_n) \in \mathbb{R}^n$, $z = (z_1, ..., z_n)$ then $f(z) = f(z_1, ..., z_n) = u(x, y)+iv(x, y)$. By Hartogs Theorem, we have: $f(z)$ is holomorphic function if and only if its real-part and imaginary-part are multi-harmonic function and conjugate each other by Cauchy-Riemann in each variable:

$$\begin{cases} \dfrac{\partial u}{\partial x_k} - \dfrac{\partial v}{\partial y_k} = 0 \\ \dfrac{\partial u}{\partial y_k} + \dfrac{\partial v}{\partial x_k} = 0, \end{cases} \tag{3}$$

where $k = 1, 2, ..., n$. The condition (3) equivalent with the "aréolaire derivative" respect to z_k be annul, that means:

$$\frac{\partial f}{\partial \overline{z}_k} = 0, \; k = 1, 2, ..., n.$$

When extending the Cauchy-Riemann system in the direction of increasing the number of functions, equations and variables, the classical methods of complex analysis does not useful in general. So, there are many authors had tried to find different ways to expand appropriately for each specific case.

The peoples who have started this way are Theodorescu and Moisil (see [3a] and [6b,c]). In around 1930-1931, they are investigated the following first-order elliptic system:

$$\begin{cases} \dfrac{\partial u_2}{\partial x_1} + \dfrac{\partial u_3}{\partial x_2} + \dfrac{\partial u_4}{\partial x_3} = 0 \\ \dfrac{\partial u_1}{\partial x_1} - \dfrac{\partial u_3}{\partial x_3} + \dfrac{\partial u_4}{\partial x_2} = 0 \\ \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_3} - \dfrac{\partial u_4}{\partial x_1} = 0 \\ \dfrac{\partial u_1}{\partial x_3} - \dfrac{\partial u_2}{\partial x_2} + \dfrac{\partial u_3}{\partial x_1} = 0 \end{cases} \tag{4}$$

To definition the "aréolaire derivative", Moisil had presented by matrix:

$$D(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}) = \begin{pmatrix} 0 & \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ \dfrac{\partial}{\partial x_1} & 0 & \dfrac{-\partial}{\partial x_3} & \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} & 0 & \dfrac{-\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_3} & \dfrac{-\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} & 0 \end{pmatrix}$$

A column-vector $u = (u_1, u_2, u_3, u_4)$ to be called holomorphic-vector in $\Omega \subset \mathbb{R}^3$ if its "aréolaire derivative" equal to zero in Ω, that means:

$$D(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})u = 0. \tag{5}$$

Theodorescu and Moisil have been presented the integral and Cauchy formula for those holomorphic-vectors. Bisatze (see [21a,b]) has already investigated the Cauchy integral type, and then, its continue by Huan-Le-Dy ([7]) and Neldelcu-Coroi ([8]). Note that, four components of the holomorphic-vector are harmonic functions in $\mathbb{R}^3$, which are associated together by vanishing "aréolaire derivative".

In 1964, Vinogradov (see [24]) has results namely "About a Cauchy-Riemann analogy in 4 dimensional space" by investigated following system

$$
\begin{cases}
\dfrac{\partial u_1}{\partial x_1} - \dfrac{\partial u_2}{\partial x_2} - \dfrac{\partial u_3}{\partial x_3} - \dfrac{\partial u_4}{\partial x_4} = 0\\
\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_3}{\partial x_4} + \dfrac{\partial u_4}{\partial x_3} = 0\\
\dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_2}{\partial x_4} + \dfrac{\partial u_3}{\partial x_1} - \dfrac{\partial u_4}{\partial x_2} = 0\\
\dfrac{\partial u_1}{\partial x_4} - \dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2} + \dfrac{\partial u_4}{\partial x_1} = 0
\end{cases} \tag{6}
$$

In this system, there are 4 equations with 4 variables, but it is considered in $\mathbb{R}^4$, Vinogradov had the same result as the result in system (4). In order to have a holomorphic-vector, we must also have four harmonic functions in $\mathbb{R}^4$ which associated together by vanishing "aréolaire derivative". In particularly, since the number of variables is 4, so from the investigation of the system (6), Vinogradov had some "unexpected results" in the holomorphic function respect to 2 complex variables.
With the addition of the number of equations, functions and variables, there are many new difficulties appear, one suggested an alternative extension: to contructed the theory of hyper-complex numbers and hyper-complex functions. Started by Moisil (see [6b, c]) in 1931, this theory has been growing steadily and has many important applications using the results of Moisil ([6]), Theodorescu ([3]), Nef ([9]), Sobrero ([10]), Fueter ([11]), Iftimie ([1]), Delanghe ([2]), Goldschmidt ([12]), Gilbert ([13]), Colton ([14]), Sommen ([15]), Brackx ([15]), etc.
Suppose $\mathcal{A}$ be a Clifford algebras which has 2^n-dimensional with basis elements $e_0, e_1, ..., e_n, e_1e_2, ..., e_{n-1}e_n, ..., e_1e_2...e_n$. Each basic vector in $\mathcal{A}$ can be presented by

$$e_A = e_{k_1}e_{k_2}...e_{k_t}$$

where

$$A = \{k_1, k_2, ..., k_t\},\ 1 \le k_1 < k_2 < ..., k_t \le n,$$

and $e_0 = e_\phi$.
If $a \in \mathcal{A}$ then a can be written as $a = \sum\limits_A a_A e_A,\ a_A \in \mathbb{R}$. When $n = 1$, we have $\mathcal{A} \equiv \mathbb{C}$, when $n = 2$, A be Quaternion algebra.
Consider a mapping

$$f : \Omega \subset \mathbb{R}^n \longrightarrow \mathcal{A}.$$

For any $x \in \Omega$, we have $f(x) = \sum\limits_A f_A(x)e_A \in \mathcal{A}$.
From basis elements $e_1, e_2, ..., e_n$ we introduce diferential operator

$$D = e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} + ... + e_n\frac{\partial}{\partial x_n}. \tag{7}$$

Some time we can also introduce the operator

$$\mu = e_0 \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + ... + e_n \frac{\partial}{\partial x_n}.$$

And then we make a following “aréolaire derivative”:

$$D(f) = \sum_{i=1}^{n} \sum_{A} e_i e_A \frac{\partial f_A}{\partial x_i}.$$

A function $f \in C^1(\Omega)$ to be called a regular in Ω if $D(f) = 0$ in Ω. Note that, if f is regular function in Ω then all components $f_A(x)$ is harmonic function in Ω. Therefore, if we have a regular function in Clifford algebras $\mathcal{A}$ which has 2^n-dimensional, we need 2^n components $f_A(x)$ whose associated together by vanishing “aréolaire derivative”. Most of results for holomorphic function can applied for regular function, and its also true when we consider about Taylor series, Laurent series, etc.

In middle in 20th century, Wekua had expanded the theory of holomorphic in another way: generalized analytic theory (see [20]). He had proven that for any first order linear elliptic system (with 2 equations and 2 variables) can prescribed by following canonical system

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} + au + bv = f \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + cu + dv = g. \end{cases} \tag{8}$$

Denoting $W = u + iv$, $A = \frac{1}{4}(a - d + ic + ib)$, $B = \frac{1}{4}(a + d + ic - ib)$; $F = \frac{1}{2}(f + ig)$, $z = x + iy$, the system (8) equivalent the following equation

$$\frac{\partial W}{\partial \overline{z}} + AW + B\overline{W} = F. \tag{9}$$

If $f = g = 0$ then we have homogeneous equations and (9) can be presented by

$$\frac{\partial W}{\partial \overline{z}} + AW + B\overline{W} = 0 \tag{10}$$

where $A, B, F \in L_p,\ p > 2$.

Vekua had considered “aréolaire derivative” $\frac{\partial W}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial W}{\partial x} + i\frac{\partial W}{\partial y})$ in distributional sense, i.e. Sobolev derivative.

Solution of equation (10) to be called “generalized analytic” function, its has properties similar holomorphic function. The theory of “generalized analytic” function can be applied to solved many problems, i.e. geometry, mechanics. Vekua and Bers (see [19]) are the pioneers for this research direction and achieve basic results. Many mathematican used Vekua's method to constructed the theory of analysis function repect to several variables by investigating following system

$$\frac{\partial W}{\partial \bar{z}_j} = A_j W + B_j \overline{W},$$

$j = 1, 2, ..., n$. Among these authors can be mentioned are Koohara ([16]), Mikhailov ([22]), Palamodov ([23]), Tutschke ([17]),...
In 1982, we had already given method to expanded the Cauchy-Riemann in several dimensional space. This method, one hand unifies the various extension methods so far for the C-R system into a common scheme, on the other hand, also creates new generalized results. These results are presented in 3 chapters:
Chapter 1. Holomorphic vectors in m-dimensional Euclidean space.
Chapter 2. Holomorphic functions taking values in Clifford algebras.
Chapter 3. A classes of first order elliptic system (extension of Vekua method).

Function Spaces

$C^k(\Omega)$:	The space of the k-order continuously differentiable functions in $Omega$, $k \geq 0$ is integer number.
$C^k(\Omega, \mathcal{A})$	$= \left\{ f : \Omega \longrightarrow \mathcal{A} \mid f = \sum_A f_A(x) e_A ; f_A \in C^k(\Omega) \right\}$
$C^{k,N}(\Omega)$:	The space of the column-vectors which has N components, such that, each component belong to $C^k(\Omega)$.
$C_0^k(\Omega)$:	The space of the k-order continuously differentiable functions and which has compact support in Ω.
$C_0^\infty(\Omega)$:	The space of the arbitrary-order continuously differentiable functions and which has compact support in Ω.
$C_\alpha(\Omega)$:	The space of the Hölder continuously functions in Ω, $0 < \alpha \leq 1$.
$C_{\alpha,N}(\Omega)$:	The space of the column-vectors which has N components, such that, each component belong to $C_\alpha(\Omega)$.
$H(\Omega)$:	Set of holomorphic vectors in Ω.
$\mathcal{H}(\Omega)$:	Set of holomorphic (hyper-complex) functions in Ω.
$(\Omega)\mathcal{H}^*$:	Set of dual holomorphic (hyper-complex) functions in Ω.
$L_{p,N}(\Omega)$: $(p \geq 1)$	The Banach space of the column-vectors which has N components, such that, each component belong to $L_p(\Omega)$ with following norm $\|\|f\|\|_{L_{p,N}(\Omega)} = \|\|f_1\|\|_{L_p(\Omega)} + ... + \|\|f_N\|\|_{L_p(\Omega)}$.
$L_{p,N}^{loc}(\Omega)$: $(p \geq 1)$	The space of the column-vectors which has N components, such that, each component belong to $L_p^{loc}(\Omega)$.
$L_{p,N^2}(\Omega)$: $(p \geq 1)$	The Banach space of the N-order square matrixs which has N components, such that, each component belong to $L_p(\Omega)$ with following norm $\|\|f\|\|_{L_{p,N^2}(\Omega)} = \sum_{i=1}^{N} \sum_{j=1}^{N} \|\|f_{ij}\|\|_{L_p(\Omega)}$.

Chapter 1

Holomorphic Vectors in m-dimensional Euclidean Space

1.1 Matrix $\tilde{D}(1, 2, ..., m; N)$ classes

A holomorphic function $f(z) = u + iv$ can equivalent to a vector which has 2 components in $\mathbb{R}^2$. Its components satifies linear first-order homogeneous partial equation (Cauchy-Riemann system)

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = 0 \end{cases} \tag{1.1}$$

In ([21]), A. V. Bisatze had prescribed "Cauchy-Riemann type in 3-dimensional Euclidean space" by consider following system

$$\begin{cases} \dfrac{\partial q_2}{\partial x_1} + \dfrac{\partial q_3}{\partial x_2} + \dfrac{\partial q_4}{\partial x_3} = 0 \\ \dfrac{\partial q_1}{\partial x_1} - \dfrac{\partial q_3}{\partial x_3} + \dfrac{\partial q_4}{\partial x_2} = 0 \\ \dfrac{\partial q_1}{\partial x_2} + \dfrac{\partial q_2}{\partial x_3} - \dfrac{\partial q_4}{\partial x_1} = 0 \\ \dfrac{\partial q_1}{\partial x_3} - \dfrac{\partial q_2}{\partial x_2} + \dfrac{\partial q_3}{\partial x_1} = 0 \end{cases} \tag{1.2}$$

(Moisil-Theodorescu system). Since the properties of the solution in Cauchy-Riemann system is also true with Moisil-Theodorescu system, then the vector $q = (q_1, q_2, q_3, q_4)$ satisfying Moisil-Theodorescu system can be called "holomorphic vector" in $\mathbb{R}^3$.

In 1964, in [24], V-S.Vinogradov had represented "Cauchy-Riemann type in 4-dimensional space" by investigated following system

$$\begin{cases} \dfrac{\partial u_1}{\partial x_1} - \dfrac{\partial u_2}{\partial x_2} - \dfrac{\partial u_3}{\partial x_3} - \dfrac{\partial u_4}{\partial x_4} = 0 \\ \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_3}{\partial x_4} + \dfrac{\partial u_4}{\partial x_3} = 0 \\ \dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_2}{\partial x_4} + \dfrac{\partial u_3}{\partial x_1} - \dfrac{\partial u_4}{\partial x_2} = 0 \\ \dfrac{\partial u_1}{\partial x_4} - \dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2} + \dfrac{\partial u_4}{\partial x_1} = 0 \end{cases} \tag{1.3}$$

For each solution of (1.3) has 4 components (u_1, u_2, u_3, u_4), its has the same properties of the solution of Cauchy-Riemann system, so it to be called "holomorphic vector" in $\mathbb{R}^4$.
There are many results which has expanded Cauchy-Riemann system in different ways (see [1], [2], [12], [15])...

In order to unify many different ways of extending that Cauchy-Riemann system into a common direction, consistently presenting the same method, and further can be expanded and generalized, we have a basic comment. As follows: each of these systems is associated with a square matrix of matrix $\tilde{D}(1, 2, ..., m; N)$ which we will be defined as following (see definition 1.1).

For instance, Moisil-Theodorescu system can be prescribed by following matrix

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & -3 & 2 \\ 2 & 3 & 0 & -1 \\ 3 & -2 & 1 & 0 \end{pmatrix} \tag{1.4}$$

The number 3 at position which has 3-line and 2-column prescribe that, in the 3th-equations, derivative of 2th-component respect to x_3 which has coeffcient equal to +1, the number "-1" at position which has 3-line and 4-column prescribe that, in the 3th-equations, derivative of 4th-component respect to x_1 which has coeffcient equal to -1. (For equations (1.1), (1.2), (1.4) and the system can be investigated in [1], [2], ['12],..., all of the coeffcients of the derivatives equal to 1 or -1).
By that denoting, the Cauchy-Riemann system can be presented by following matrix

$$\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \tag{1.5}$$